Chapter 5

MATHEMATICAL INDUCTION

In mathematics, as in science, there are two general methods by which we can arrive at new results. One, deduction, involves the assumption of a set of axioms from which we deduce other statements, called theorems, according to prescribed rules of logic. This method is essentially that used in standard courses in Euclidean geometry.

The second method, induction, involves the guessing or discovery of general patterns from observed data. While in most branches of science and mathematics the guesses based on induction may remain merely conjectures, with varying degrees of probability of correctness, certain conjectures in mathematics which involve the integers frequently can be proved by a technique of Pascal called mathematical induction. Actually, this technique in not induction, but is rather an aid in proving conjectures arrived at by induction.

THE PRINCIPLE OF MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all positive integers if (a) it is true for 1, and (b) its being true for any integer k implies that it is true for the next integer k + 1.*

If one replaces (a) by (a'),"it is true for some integer s," then (a') and (b) prove the statement true for all integers greater than or equal to s. Part (a) gives only a starting point; this starting point may be any integer - positive, negative, or zero.

Let us see if mathematical induction is a reasonable method of proof of a statement involving integers n. Part (a) tells us that the statement is true for $n = 1$. Using (b) and the fact that the statement is true for 1, we obtain the fact that it is true for the next integer 2. Then (b) implies that it is true for $2 + 1 = 3$. Continuing in this way, we would ultimately reach any fixed positive integer.

Let us use this approach on the problem of determining a formula which will give us the number of diagonals of a convex polygon in terms of the number of sides. The three-sided polygon, the triangle, has no diagonals; the four-sided polygon has two. An examination of other cases yields the data included in the following table:

n = number of sides	3	4	5	6	7	8	9	...	n	...
D_n = number of diagonals	0	2	5	9	14	20	27	...	D_n	...

The task of guessing the formula, if a formula exists, is not necessarily an easy one, and there is no sure approach to this part of the over-all problem. However, if one is perspicacious, one observes the following pattern:

$$
\begin{aligned}
2D_3 &= 0 = 3\cdot 0\\
2D_4 &= 4 = 4\cdot 1\\
2D_5 &= 10 = 5\cdot 2\\
2D_6 &= 18 = 6\cdot 3\\
2D_7 &= 28 = 7\cdot 4.
\end{aligned}
$$

This leads us to conjecture that

$$2D_n = n(n - 3)$$

or

$$D_n = \frac{n(n-3)}{2}$$

Now we shall use mathematical induction to prove this formula. We shall use as a starting point $n = 3$, since for n less than 3 no polygon exists. It is clear from the data that the formula holds for the case $n = 3$. Now we assume that a k-sided polygon has $k(k - 3)/2$ diagonals. If we can conclude from this that a $(k + 1)$-sided polygon has $(k + 1)[(k + 1) - 3]/2 = (k + 1)(k - 2)/2$ diagonals, we will have proved that the formula holds for all positive integers greater than or equal to 3.

Consider a k-sided polygon. By assumption it has $k(k - 3)/2$ diagonals. If we place a triangle on a side AB of the polygon, we make it into a $(k + 1)$-sided polygon. It has all the diagonals of the k-sided polygon plus the diagonals drawn from the new vertex N to all the

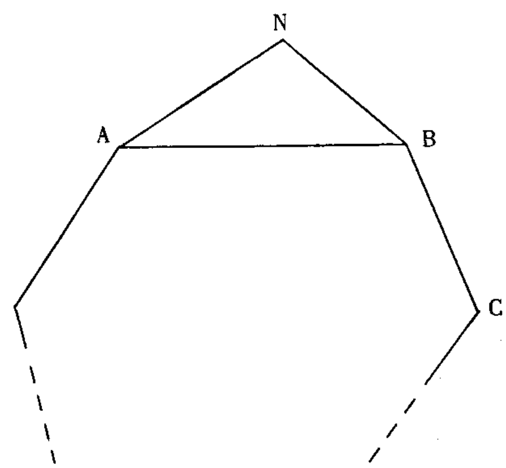

Figure 3

vertices of the previous k-sided polygon except 2, namely A and B. In addition, the former side AB has become a diagonal of the new $(k + 1)$-sided polygon. Thus a $(k + 1)$-sided polygon has a total of $\frac{k(k-3)}{2} + (k - 2) + 1$ diagonals. But:

$$\frac{k(k-3)}{2} + (k - 2) + 1$$
$$= \frac{k^2 - 3k + 2k - 2}{2}$$

$$= \frac{k^2 - k - 2}{2}$$
$$= \frac{(k + 1)(k - 2)}{2}$$
$$= \frac{(k + 1)[(k + 1) - 3]}{2}$$

This is the desired formula for $n = k + 1$.

So, by assuming that the formula $D_n = n(n - 3)/2$ is true for $n = k$, we have been able to show it true for $n = k + 1$. This, in addition to the fact that it is true for $n = 3$, proves that it is true for all integers greater than or equal to 3. (The reader may have discovered a more direct method of obtaining the above formula.)

The method of mathematical induction is based on something that may be considered one of the axioms for the positive integers: If a set S contains 1, and if, whenever S contains an integer k, S contains the next integer $k + 1$, then S contains all the positive integers. It can be shown that this is equivalent to the principle that in every non-empty set of positive integers there is a least positive integer.

Example 1. Find and prove by mathematical induction a formula for the sum of the first n cubes, that is, $1^3 + 2^3 + 3^3 + ... + n^3$.

Solution: We consider the first few cases:

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100. \end{aligned}$$

We observe that $1 = 1^2$, $9 = 3^2$, $36 = 6^2$, and $100 = 10^2$. Thus it appears that the sums are the squares of triangular numbers 1, 3, 6, 10, ... In Chapter 4 we saw that the triangular numbers are of the form $n(n + 1)/2$. This suggests that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n + 1)}{2}\right]^2.$$

It is clearly true for $n = 1$. Now we assume that it is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k + 1)}{2}\right]^2.$$

Can we conclude from this that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2 ?$$

We can add $(k + 1)^3$ to both sides of the known expression, obtaining:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= (k+1)^2 \frac{k^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2 (k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2 (k+2)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2 \\
&= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2 .
\end{aligned}$$

Hence the sum when $n = k + 1$ is $[n(n + 1)/2]^2$, with n replaced by $k + 1$, and the formula is proved for all positive integers n.

Our guessed expression for the sum was a fortunate one!

Example2. Prove that $a - b$ is a factor of $a^n - b^n$ for all positive integers n.

Proof: Clearly, $a - b$ is a factor of $a^1 - b^1$; hence the first part of the induction is verified, that is, the statement is true for $n = 1$. Now we assume that $a^k - b^k$ has $a - b$ as a factor:

$$a^k - b^k = (a - b)M.$$

Next we must show that $a - b$ is a factor of $a^{k+1} - b^{k+1}$. But

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a \cdot a^k - b \cdot b^k \\
&= a \cdot a^k - b \cdot a^k + b \cdot a^k - b \cdot b^k \\
&= (a - b)a^k + b(a^k - b^k).
\end{aligned}$$

Now, using the assumption that $a^k - b^k = (a - b)M$ and substituting, we obtain:

$$\begin{aligned} a^{k+1} - b^{k+1} &= (a - b)a^k + b(a - b)M \\ &= (a - b)\ [a^k + bM]. \end{aligned}$$

We see from this that $a - b$ is a factor of $a^{k+1} - b^{k+1}$ and hence $a - b$ is a factor of $a^n - b^n$ for n equal to any positive integer. It is easily seen that the explicit factorization is

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}).$$

Example 3. Prove that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n, that is, there is an integer u such that $n(n^2 + 5) = 6$u.

Proof: We begin by proving the desired result for all the integers greater than or equal to 0 by mathematical induction.

When $n = 0$, $n(n^2 + 5)$ is 0. Since $0 = 6 \cdot 0$ is a multiple of 6, the result holds for $n = 0$.

We now assume it true for $n = k$, and seek to derive from this its truth for $n = k + 1$. Hence we assume that

$$(1) \qquad k(k^2 + 5) = 6r$$

with r an integer. We then wish to show that

$$(2) \qquad (k + 1)[(k + 1)^2 + 5] = 6s$$

with s an integer. Simplifying the difference between the left-hand sides of (2) and (1), we obtain

$$(3) \qquad (k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 3k(k + 1) + 6.$$

Since k and $k + 1$ are consecutive integers, one of them is even. Then their product $k(k + 1)$ is even, and may be written as $2t$, with t an integer. Now (3) becomes

$$(4) \qquad (k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 6t + 6 = 6(t + 1).$$

Transposing, we have

$$(k + 1)[(k + 1)^2 + 5] = k(k^2 + 5) + 6(t + 1).$$

Using (1), we can substitute $6r$ for $k(k^2 + 5)$. Hence

$$(k + 1)[(k + 1)^2 + 5] = 6r + 6(t + 1) = 6(r + t + 1).$$

Letting s be the integer $r + t + l$, we establish (2), which is the desired result when $n = k + 1$. This completes the induction and proves the statement for $n \geq 0$.

Now let n be a negative integer, that is, let $n = -m$, with m a positive integer. The previous part of the proof shows that $m(m^2 + 5)$ is of the form $6q$, with q an integer. Then

$$n(n^2 + 5) = (-m)[(-m)^2 + 5] = -m(m^2 + 5) = -6q = 6(-q),$$

a multiple of 6. The proof is now complete.

We have seen that binomial coefficients, Fibonacci and Lucas numbers, and factorials may be defined inductively, that is, by giving their initial values and describing how to get new values from previous values. Similarly, one may define an arithmetic progression $a_1, a_2, \dots, a_t$ as one for which there is a fixed number d such that $a_{n+1} = a_n + d$ for $n = 1, 2, \dots, t - 1$. Then the values of a_1 and d would determine the values of all the terms. A geometric progression $b_1, \dots, b_t$ is one for which there is a fixed number r such that $b_{n+1} = b_n r$ for $n = 1, 2, \dots, t - 1$; its terms are determined by b_1 and r.

It is not surprising that mathematical induction is very useful in proving results concerning quantities that are defined inductively, however, it is sometimes necessary or convenient to use an alternate principle, called **strong mathematical induction**.

STRONG MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all the positive integers if there is an integer q such that (a) the statement is true for 1, 2, ... , q, and (b) when $k \geq q$, the statement being true for 1, 2, ... , k implies that it is true for k + 1.*

As in the case of the previous principle, this can be modified to apply to statements in which the starting value is an integer different from 1.

We illustrate strong induction in the following:

Example 4. Let $a, b, c, r, s,$ and t be fixed integers. Let $L_0, L_1, \dots$ be the Lucas sequence. Prove that

$$\text{(A)} \qquad rL_{n+a} = sL_{n+b} + tL_{n+c}$$

is true for $n = 0, 1, 2, \dots$ if it is true for $n = 0$ and $n = 1$.

Proof: We use strong induction. It is given that (A) is true for $n = 0$ and $n = 1$. Hence, it remains to assume that $k \geq 1$ and that (A) is true for $n = 0, 1, 2, \dots, k$, and to use these assumptions to prove that (A) holds for $n = k + 1$.

We therefore assume that

$$\begin{aligned} rL_a &= sL_b + tL_c \\ rL_{1+a} &= sL_{1+b} + tL_{1+c} \\ rL_{2+a} &= sL_{2+b} + tL_{2+c} \\ &\cdots \\ rL_{k-1+a} &= sL_{k-1+b} + tL_{k-1+c} \\ rL_{k+a} &= sL_{k+b} + L_{k+c} \end{aligned}$$

and that there are at least two equations in this list. Adding corresponding sides of the last two of these equations and combining like terms, we obtain

$$r(L_{k+a} + L_{k-1+a}) = s(L_{k+b} + L_{k-1+b}) + t(L_{k+c} + L_{k-1+c}).$$

Using the relation $L_{n+1} + L_n = L_{n+2}$ for the Lucas numbers, this becomes

$$rL_{k+1+a} = sL_{k+1+b} + tL_{k+1+c}$$

which is (A) when $n = k + 1$. This completes the proof.

Problems for Chapter 5

In Problems 1 to 10 below, use mathematical induction to prove each statement true for all positive integers n.

1. The sum of the interior angles of a convex $(n + 2)$-sided polygon is $180n$ degrees.

2. $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$.

3. (a) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = n(4n^2 - 1)/3$.

   (b) $1{\cdot}3 + 3{\cdot}5 + 5{\cdot}7 + \ldots + (2n - 1)(2n + 1) = n(4n^2 + 6n - 1)/3$.

   (c) $\dfrac{1}{1{\cdot}3} + \dfrac{1}{2{\cdot}4} + \dfrac{1}{3{\cdot}5} + \ldots + \dfrac{1}{n(n + 2)} = \dfrac{n(3n + 5)}{4(n + 1)(n + 2)}$.

   (d) $1 + 2a + 3a^2 + \ldots + na^{n-1} = [1 - (n + 1)a^n + na^{n+1}]/(1 - a)^2$.

4. (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$.

   (b) $1{\cdot}3 + 2{\cdot}4 + 3{\cdot}5 + \ldots + n(n + 2) = n(n + 1)(2n + 7)/6$.

   (c) $\dfrac{5}{1{\cdot}2}{\cdot}\dfrac{1}{3} + \dfrac{7}{2{\cdot}3}{\cdot}\dfrac{1}{3^2} + \dfrac{9}{3{\cdot}4}{\cdot}\dfrac{1}{3^3} + \ldots + \dfrac{2n + 3}{n(n + 1)}{\cdot}\dfrac{1}{3^n} = 1 - \dfrac{1}{3^n(n + 1)}$.

5. $(1^3 + 2^3 + 3^3 + \ldots + n^3) + 3(1^5 + 2^5 + 3^5 + \ldots + n^5) = 4(1 + 2 + 3 + \ldots + n)^3$.

6. $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1 + 2 + 3 + \ldots + n)^4$

*7. $3^n + 7^n - 2$ is an integral multiple of 8.

*8. $2 \cdot 7^n + 3 \cdot 5^n - 5$ is an integral multiple of 24.

9. $x^{2n} - y^{2n}$ has $x + y$ as a factor.

10. $x^{2n+1} + y^{2n+1}$ has $x + y$ as a factor.

11. For all integers n, prove the following:

(a) $2n^3 + 3n^2 + n$ is an integral multiple of 6.
(b) $n^5 - 5n^3 + 4n$ is an integral multiple of 120.

12. Prove that $n(n^2 - 1)(3n + 2)$ is an integral multiple of 24 for all integers n.

13. Guess a formula for each of the following and prove it by mathematical induction:

(a) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n(n+1)}$.

(b) $(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)\ldots(x^{2^n} + y^{2^n})$.

14. Guess a formula for each of the following and prove it by mathematical induction:

(a) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)$.

(b) $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \ldots + \dfrac{1}{(2n-1)(2n+1)}$.

15. Guess a simple expression for the following and prove it by mathematical induction:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right).$$

16. Find a simple expression for the product in Problem 15, using the factorization

$$x^2 - y^2 = (x - y)(x + y).$$

17. Prove the following properties of the Fibonacci numbers F_n for all integers n greater than or equal to 0:

(a) $2(F_s + F_{s+3} + F_{s+6} + \dots + F_{s+3n}) = F_{s+3n+2} - F_{s-1}$.

(b) $F_{-n} = (-1)^{n+1}F_n$.

(c) $\binom{n}{0}F_s + \binom{n}{1}F_{s+1} + \binom{n}{2}F_{s+2} + \dots + \binom{n}{n}F_{s+n} = F_{s+2n}$.

18. Discover and prove formulas similar to those of Problem 17 for the Lucas numbers L_n.

19. Use Example 4, in the text above, to prove the following properties of the Lucas numbers for $n = 0, 1, 2, \dots$, and then prove them for all negative integers n.

(a) $L_{n+4} = 3L_{n+2} - L_n$.

(b) $L_{n+6} = 4L_{n+3} + L_n$.

(c) $L_{n+8} = 7L_{n+4} - L_n$.

(d) $L_{n+10} = 11L_{n+5} + L_n$.

20. State an analogue of Example 4 for the Fibonacci numbers instead of the Lucas numbers and use it to prove analogues of the formulas of Problem 19.

21. In each of the following parts, evaluate the expression for some small values of n, use this data to make a conjecture, and then prove the conjecture true for all integers n.

(a) $F_{n+1}^2 - F_n F_{n+2}$.

(b) $\dfrac{F_{n+2}^2 - F_{n+1}^2}{F_n}$.

(c) $F_{n-1} + F_{n+1}$.

22. Discover and prove formulas similar to the first two parts of the previous problem for the Lucas numbers.

23. Prove the following for all integers m and n:

(a) $L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$.

(b) $F_{m+n+1} = F_{m+1}F_{n+1} - F_mF_n$.

24. Prove that $(F_{n+1})^2 + (F_n)^2 = F_{2n+1}$ for all integers n.

25. Let a and b be the roots of the quadratic equation $x^2 - x - 1 = 0$. Prove that:

(a) $F_n = \dfrac{a^n - b^n}{a - b}$.

(b) $L_n = a^n + b^n$.

(c) $F_n L_n = F_{2n}$.

(d) $a^n = aF_n + F_{n-1}$ and $b^n = bF_n + F_{n-1}$.

26. The sequence 0, 1, ½, 3/4, 5/8, 11/16, ... is defined by

$$u_0 = 0,\ u_1 = 1,\ u_2 = \frac{u_1 + u_0}{2},\ \ldots,\ u_{n+2} = \frac{u_{n+1} + u_n}{2},\ \ldots\ .$$

Discover and prove a compact formula for u_n as a function of n.

27. The Pell sequence 0, 1, 2, 5, 12, 29, ... is defined by

$$P_0 = 0,\ P_1 = 1,\ P_2 = 2P_1 + P_0,\ \ldots,\ P_{n+2} = 2P_{n+1} + P_n,\ \ldots\ .$$

Let $x_n = P_{n+1}^2 - P_n^2$, $y_n = 2P_{n+1}P_n$, and $z_n = P_{n+1}^2 + P_n^2$. Prove that for every positive integer n the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers.

28. Discover and prove properties of the Pell sequence that are analogous to those of the Fibonacci sequence.

29. Let the sequence 1, 5, 85, 21845, ... be defined by

$$c_1 = 1,\ c_2 = c_1(3c_1 + 2),\ \ldots,\ c_{n+1} = c_n(3c_n + 2),\ \ldots\ .$$

Prove that $c_n = \dfrac{4^{2^{n-1}} - 1}{3}$ for all positive integers n.

30. Let a sequence be defined by $d_1 = 4$, $d_2 = (d_1)^2$, ..., $d_{n+1} = (d_n)^2$, ...

Show that $d_n = 3c_n + 1$, where c_n is as defined in the previous problem.

31. Prove that $1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + \ldots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

*32. Certain of the above formulas suggest the following:

$$1\cdot 2\cdots m + 2\cdot 3\cdots(m+1) + \cdots + n(n+1)\cdots(n+m-1) = \frac{n(n+1)\ldots(n+m)}{m+1}.$$

Prove it for general m.

*33. Prove that $n^5 - n$ is an integral multiple of 30 for all integers n.

*34. Prove that $n^7 - n$ is an integral multiple of 42 for all integers n.

*35. Show that every integer from 1 to $2^{n+1} - 1$ is expressible uniquely as a sum of distinct powers of 2 chosen from 1, 2, 2^2, ... , 2^n.

*36. Show that every integer s from $-\frac{3^{n+1} - 1}{2}$ to $\frac{3^{n+1} - 1}{2}$ has a unique expression of the form

$$s = c_0 + 3c_1 + 3^2c_2 + \ldots + 3^nc_n$$

where each of c_0, c_1, ..., c_n is 0, 1, or -1.